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4.5. Other Continuous Distributions

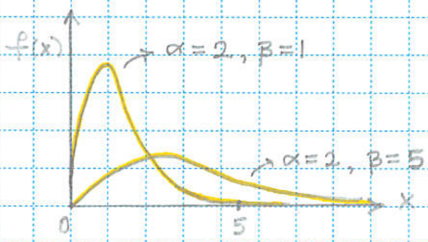
The Weibull distribution

Definition. A rv X is said to have Weibull distribution with parameters α and β ($\alpha>0$, $\beta>0$) if the pdf of X is

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x>0.$$

→ when $\alpha=1$, Weibull pdf gives the exponential distribution with $\lambda=\frac{1}{\beta}$.

→ β: scale parameter; different values stretch or compress the graph in the x-direction.



(*) $\mu=\beta\Gamma\left(1+\frac{1}{\alpha}\right)$ → mean

(*) $\sigma^2=\beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right)-\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}$ → variance

(*) $F(x;\alpha,\beta)=\begin{cases}0, & x<0\\ 1-e^{-(x/\beta)^{\alpha}}, & x\geq 0\end{cases}$ → cdf $\left(\text{obtained by } \int_0^x f(y;\alpha,\beta)\,dy\right)$

A Weibull model may not fit well when the smallest possible value of X is a nonzero constant γ, which can be treated as a third parameter in the distribution as in Weibull's original work. Namely, we would have the pdf

$$f(x;\alpha,\beta,\gamma)=\frac{\alpha}{\beta^{\alpha}}(x-\gamma)^{\alpha-1}e^{-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}}$$

and the cdf

$$F(x;\alpha,\beta,\gamma)=1-e^{-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}}, \quad x\geq 0.$$

Example. The lifetime X (in hundreds of hours) of a type of vacuum tube has a Weibull distribution with parameters $\alpha=2$ and $\beta=3$.

a) Compute $E(X)$ and $V(X)$.

b) Find $P(1.5\leq X\leq 6)$.

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c) Find the median life time of such tubes.

<u>Solution.</u> a) $E(X) = \beta\Gamma\left(1+\frac{1}{\alpha}\right) = 3\Gamma\left(1+\frac{1}{2}\right) = 3\Gamma\left(\frac{3}{2}\right) = 3\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{2} = 2.66.$

$\Gamma(n) = (n-1)\Gamma(n-1)$

$V(X) = 9\left[\Gamma\left(1+\frac{2}{2}\right) - \Gamma^2\left(1+\frac{1}{2}\right)\right] = 9\left[\Gamma(2) - \Gamma^2\left(\frac{3}{2}\right)\right] = 1.926$

b) $P(1.5 \leq X \leq 6) = \left(1-e^{-(6/3)^2}\right) - \left(1-e^{-(1.5/3)^2}\right) = e^{-.25} - e^{-4} = .760.$

c) Set $.5 = F(x) \Rightarrow .5 = 1 - e^{-(x/3)^2}$

$\Rightarrow .5 = 1 - e^{x^2/9}$

$\Rightarrow \ln(.5) = \ln\left(1 - e^{x^2/9}\right)$

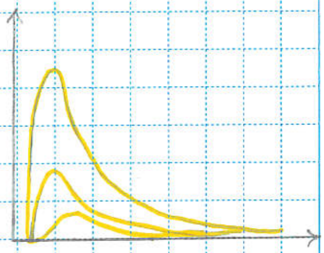
$\Rightarrow \ln(.5) = \ln 1 - \ln e^{x^2/9}$

$\Rightarrow \ln(.5) = -\frac{x^2}{9}$

$\Rightarrow x^2 = -9\ln(.5)$

$\Rightarrow x^2 = -9(-.693) (= 6.2383)$

$\Rightarrow x = 2.50.$



Lognormal density curves

<u>The lognormal distribution</u>

<u>Definition.</u> A nonnegative rv X is said to have a <u>lognormal distribution</u> if the rv $Y = \ln(X)$ has a normal distribution. The resulting pdf of a lognormal rv when $\ln(X)$ is normally distributed with parameters μ and σ is

$$f(x;\mu,\sigma) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, \quad x > 0.$$

Here, μ and σ are the mean and standard deviation of $\ln(X)$, not of X.

⊛ $E(X) = e^{\mu + \frac{\sigma^2}{2}}$

⊛ $V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$

⊛ $F(x;\mu,\sigma) = P(X \leq x) = P[\ln(X) \leq \ln(x)]$ → Because $\ln(X)$ has a normal dist., the cdf of X can be expressed in terms of $\Phi(z)$ of a standard normal rv Z.

$= P\left[\frac{\ln(X)-\mu}{\sigma} \leq \frac{\ln(x)-\mu}{\sigma}\right]$

$= P\left[Z \leq \frac{\ln(x)-\mu}{\sigma}\right] = \Phi\left[\frac{\ln(x)-\mu}{\sigma}\right].$

⊛ $F'(x;\mu,\sigma) = f(x;\mu,\sigma).$

Example. A theoretical justification based on a material failure mechanism underlies the assumption that ductile strength X of a material has a lognormal distribution. The parameters are $\mu = 5$ and $\sigma = .1$.

a) If ten different samples of an alloy steel of this type were subjected to a strength test, how many would you expect to have strength of at least 125?

b) If the smallest 5% of strength values were unacceptable, what would the minimum acceptable strength be?

Solution. a) We wish to find P(any particular one has $X > 125$). First find

$$P(X > 125) = 1 - P(X < 125) = 1 - \Phi\left(\frac{\ln(125) - 5}{.1}\right) = 1 + \Phi(-1.72) = .9573.$$

Then, the expected number is

$$10(.9573) = 9.573. \quad \text{(Ten independent samples)}.$$

b) We wish to find the 5th percentile. That is,

$$P(X < x) = P\left(Z < \frac{\ln(x) - \mu}{\sigma}\right) = .05$$

Now, remember that $\ln(x)$ has a normal distribution. Then,

$$P\left(Z \le \frac{\ln(x) - \mu}{\sigma}\right) = P(Z \le z_{.05}) = .05$$

$$\Rightarrow z_{.05} = -1.645 \quad \text{(by z table)}$$

As a result,

$$\frac{\ln(x) - \mu}{\sigma} = z_{.05}$$

$$\Rightarrow x = e^{\mu + \sigma \cdot z_{.05}}$$

$$\Rightarrow x = e^{5 + .1(-1.645)}$$

$$\Rightarrow x = 125.90.$$

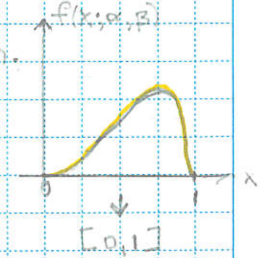
The beta distribution

The beta distribution provides positive density only for X in an interval of finite length.

Definition. A rv X is said to have a beta distribution with parameters $\alpha, \beta (>0)$, A, and B if the pdf of X is

$$f(x;\alpha,\beta,A,B) = \frac{1}{B-A}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\left(\frac{x-A}{B-A}\right)^{\alpha-1}\left(\frac{B-x}{B-A}\right)^{\beta-1}, \quad A \le x \le B.$$

When $A=0$, $B=1$, X is said to have standard beta distribution.



∇_0 The standard beta distribution is commonly used to model variation in the proportion or percentage of a quantity occurring in different samples.

⊛ $\mu = A + (B-A)\cdot\frac{\alpha}{\alpha+\beta}$

⊛ $\sigma^2 = \frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Example. What condition on α and β is necessary for the standard beta pdf to be symmetric?

Solution. If the dist. is symmetric, then $\mu = \frac{1}{2}$. Namely,

$$\frac{\alpha}{\alpha+\beta} = \frac{1}{2} \Rightarrow \alpha = \beta.$$

4.7. Transformations of a Random Variable

Often we need to deal with a transformation $Y = g(X)$ of the rv X. For instance, if we have $Y = 60X$, we first find the cdf of Y, then its pdf by differentiation. However, sometimes it is not easy to evaluate the integral in the cdf. In such situations, we use the following theorem.

Transformation Theorem. Let X have pdf $f_X(x)$ and let $Y = g(X)$, where g is monotonic (either strictly increasing or strictly decreasing) on the set of all possible values of X, so it has an inverse function $X = g^{-1}(Y) = h(Y)$. Assume that h has a derivative $h'(y)$. Then,

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| .$$

Proof. Assume that g is monotonically increasing. First, find the cdf of Y:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le h(y)) = F_X(h(y)).$$

By differentiation,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F_X(h(y))\right] = F_X'(h(y)) \cdot h'(y) = f_X(h(y)) \cdot h'(y).$$

The proof for g is monotonically decreasing can be done similarly, and the absolute value in the statement of the expression is needed in this case. The set of possible values for Y is obtained by applying g to the set of possible values for X. ∎

Example. Let X have the pdf $f_X(x) = \frac{2}{x^3}$, $x > 1$. Find the pdf of $Y = \sqrt{X}$.

Solution. We have $y = \sqrt{x} \Rightarrow x = y^2$ and $x > 1 \Rightarrow y > 1$.

Applying Transformation Theorem yields

$$\begin{aligned} f_Y(y) &= f_X(h(y)) \cdot |h'(y)| \\ &= f_X(y^2) \cdot \left|\frac{dx}{dy}\right| \\ &= f_X(y^2) \cdot |2y| \\ &= \frac{2}{(y^2)^3} \cdot 2y \\ &= \frac{4}{y^5}, \quad \text{for } y > 1. \end{aligned}$$

Example. The variation in a certain electrical current source X (in milliamps) can be modeled by the pdf

$$f_X(x) = 1.25 - .25x \quad , \; 2 \le x \le 4 .$$

If this current passes through a 220-Ω resistor, the resulting power Y (in microwatts) is given by $Y = 220X^2$. Find the pdf of Y.

Solution. $y = g(x) = 220x^2$ is not monotone on the entire line as it represents a parabola. However, it is monotonically increasing on the range of X; that is [2, 4]. Then, it has an inverse function: $x = g^{-1}(y) = h(y) = \sqrt{\dfrac{y}{220}}$. So, we can apply the Transformation Theorem,

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

$$= f_X\left(\sqrt{\frac{y}{220}}\right) \cdot \left|\frac{dx}{dy}\right| \qquad \frac{\sqrt{220}}{2\sqrt{y}} \cdot \frac{1}{\underset{\sqrt{220}\cdot\sqrt{220}}{220}} = \frac{1}{2\sqrt{220y}}$$

$$= \left(1.25 - .25\sqrt{\frac{y}{220}}\right) \cdot \frac{1}{2\sqrt{\frac{y}{220}}} \cdot \frac{1}{220}$$

$$= \left(1.25 - .25\sqrt{\frac{y}{220}}\right) \frac{1}{2\sqrt{220y}}$$

$$= \frac{125}{100} \cdot \frac{1}{2\sqrt{220y}} - \frac{25}{100}\sqrt{\frac{y}{220}} \cdot \frac{1}{2\sqrt{220y}}$$

$$= \frac{5}{8\sqrt{220y}} - \frac{1}{1760}$$

The set of possible Y values is determined by substituting $x = 2$ and $x = 4$ into $g(x) = 220x^2$. Then, the range for Y is [880, 3520].

Therefore, the pdf of $Y = 220X^2$ is

$$f_Y(y) = \begin{cases} \dfrac{5}{8\sqrt{220y}} - \dfrac{1}{1760} & , \; 880 \le y \le 3520 \\ 0 & \text{otherwise} \end{cases}$$

Example. Consider a standard normal rv Z with $Y = Z^2$. Find the pdf of Y.

Solution. Since $Y = Z^2$ is not monotonic, define $U = |Z|$. Now, Z has a symmetric distribution. Then, we can write

$$f_U(u) = f_Z(u) + f_Z(-u) = 2 f_Z(u). \quad \cdots (*)$$

Then, $Y = Z^2 = |Z|^2 = U^2$ and the transformation $Y = U^2$ is monotonic because its sets of possible values is $(0, \infty)$. Thus, we can use the Transformation Theorem with $h(y) = y^{1/2}$; that is,

$$f_Y(y) = f_U[h(y)] \cdot |h'(y)|$$

$$= 2 f_Z[h(y)] \cdot |h'(y)|$$

$$= 2 \cdot f_Z[y^{1/2}] \cdot \left|\frac{1}{2y^{1/2}}\right|$$

$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$$= \frac{2}{\sqrt{2\pi}} e^{-\frac{(y^{1/2})^2}{2}} \cdot \frac{1}{2y^{1/2}}, \quad y > 0$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}, \quad y > 0.$$

Justification for $(*)$:

$F_U(u) = P(U \le u) = P(|Z| \le u) = P(-u \le Z \le u) = 2P(0 \le Z \le u) = 2[F_Z(u) - F_Z(0)]$.

By differentiation w.r.t u, we obtain

$$f_U(u) = 2 f_X(u).$$

Example a) If a measurement error X is uniformly distributed on $[-1, 1]$, find the pdf of $Y = |X|$, which is the magnitude of the measurement error.

b) If $X \sim \text{Unif}[-1, 3]$, find the pdf of $Y = X^2$.

Solution. a) $Y = |X|$ is not monotone on $[-1, 1]$. Then, we use cdf method. Possible values of Y are $\{y : 0 \le y \le 1\}$. For $0 \le y \le 1$,

$$F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) = \frac{y - (-y)}{1 - (-1)} = y$$

by uniform cdf

By differentiation,

$$F_Y'(y) = 1 = f_Y(y).$$

b) The transformation $y = x^2$ is not monotone and is not symmetric on $[-1, 3]$. Then, we have to consider two cases. We have $0 \le Y \le 9$.

* For $0 \le y \le 1$,

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \frac{\sqrt{y} - (-\sqrt{y})}{3-(-1)} = \frac{\sqrt{y}}{2}.$$

* For $1 \le y \le 9$,

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-1 \le X \le 1 \text{ or } 1 \le X \le \sqrt{y}) = \frac{1}{2} + \frac{\sqrt{y}-1}{3-(-1)} = \frac{1+\sqrt{y}}{4}.$$

$$\frac{1-(-1)}{3-(-1)} = \frac{1}{2}.$$

By differentiation, we have

$$f_Y(y) = \begin{cases} \dfrac{1}{4\sqrt{y}} & , 0 < y \le 1 \\ \dfrac{1}{8\sqrt{y}} & , 1 < y \le 9 \\ 0 & \text{otherwise} \end{cases}$$