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2.3 Counting Techniques

If all outcomes are equally likely, probabilities are found by counting.

The product rule for ordered pairs

Here, we wish to count the ordered pairs by applying our first counting rule. In an ordered pair, (O_1, O_2), order matters. That is, $(O_1, O_2) \neq (O_2, O_1)$.

Proposition. If the first element of an ordered pair can be selected in n_1 ways and the second in n_2 ways, then there are $n_1 n_2$ ordered pairs.

We will refer to this proposition as the product rule. Sometimes, it is stated as multiplication rule.

Example. A homeowner needs to hire one plumbing contractor and one electrical contractor. There are 4 plumbing contractors and 3 electrical contractors available. In how many ways can the two contractors be chosen?

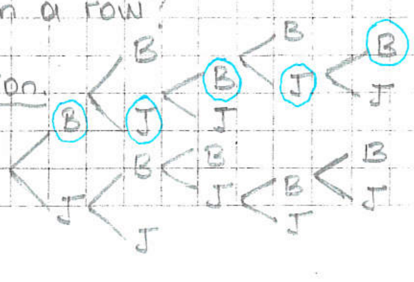
Solution. Let the plumbers be P_1, P_2, P_3, P_4 and the electricians be E_1, E_2, E_3. Each choice is a pair (P_i, E_j) with $i \in \{1, \dots, 4\}$ and $j \in \{1, \dots, 3\}$. By the product rule, $N = 4 \cdot 3 = 12$.

Tree diagrams

Tree diagrams are a useful way to represent possibilities in counting problems.

Example. Bill and John keep playing chess until one of them wins two games in a row or three games altogether. In what percent of all possible cases does the game end because Bill wins three games without winning two in a row?

Solution.



The total number of possible cases is equal to the number of the endpoints of the branches, which is 10. There is only one sequence, BJBJB, where Bill reaches 3 wins without winning two in a row. Then, the answer is 10%.

A more general product rule

We call an ordered collection of k objects a k-tuple. The product rule for k-tuples is given below.

Proposition. Suppose a set consists of ordered collections of k elements (k-tuples). If the first element can be chosen in n_1 ways, the second in n_2 ways, ..., and the kth in n_k ways, then there are $n_1 n_2 \dots n_k$ possible k-tuples.

Example. A homeowner is remodeling their kitchen. They need to buy an appliance from one of 5 dealers and hire one plumbing contractor from 4 available and one electrical contractor from 3 available.

Solution. Let $D_1, \dots, D_5$ be the dealers, $P_1, \dots, P_4$ be the plumbing contractors, and $E_1, \dots, E_3$ be the electrical contractors. Then, each choice can be represented as a 3-tuple (D_i, P_j, E_k). By the product rule for k-tuples, we obtain

$$N = n_1 n_2 n_3 = 5.4.3 = 60.$$

Thus, 60 ordered choices (3-tuple) are possible.

Permutations

So far k-tuple elements come from different sets, such as dealers, plumbing contractors, and electrical contractors, with replacement (the same element can appear more than once). Now, we consider a k-tuple from a fixed set of n elements without replacement (each element appears at most once).

Definition. An ordered sequence of k objects selected from a set of n distinct objects is called a permutation of size k, and is denoted by $P_{k,n}$.

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Definition. For any positive integer m, $m! = m(m-1)\dots 2.1$, which is called m factorial. Also, $0! = 1$.

Now, we can define the permutation as

$$P_{k,n} = n(n-1)\dots(n-(k-2))(n-(k-1))$$
$$= n(n-1)\dots(n-k+2)(n-k+1)\frac{(n-k)!}{(n-k)!}$$
$$= \frac{n!}{(n-k)!}$$

Example. An exam has four questions, and each must be graded by a different assistant. Ten teaching assistans are available. In how many ways can the assistans be assigned?

Solution. Let n be the number of assistants, and k be the number of questions. Then, we have

$$P_{4,10} = \frac{10!}{(10-4)!} = \frac{10.9.8.7.6!}{6!} = 5040.$$

Combinations

Definition. Given a set of n distinct objects, any unordered subset of size k of the objects is called a combination, and is denoted by $\binom{n}{k}$.

The number of combinations is smaller than the number of permutations because order does not matter. We define the combination as

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}.$$

Here, a combination is found by dividing permutations by $k!$, since different orders of the same objects count as one.

(*) $\binom{n}{n} = 1 \rightarrow$ there is only one way to choose all n elements from a set of n.

(*) $\binom{n}{0} = 1 \rightarrow$ there is only one way to choose no elements – empty set.

(*) $\binom{n}{1} = n \rightarrow$ there are n ways to choose a single element from n elements.

Example. A random sample of 45 students was asked if they are happy with their instructors. 32 students responded negatively. What is the probability that Mehmet, Nehir, and Yunus all gave negative responses?

Solution. ⊛ Total students: 45

⊛ Number of negative responses: 32

Step 1. Choosing 32 negative responders from 45 students: $\binom{45}{32}$

Step 2. Choosing the remaining 29 negative responders: $\binom{42}{29}$

Step 3. Probability that Mehmet, Nehir, and Yunus all gave negative responses:

Since each event of 32 students is equally likely, the probability is

$$P(A) = \frac{N(A)}{N} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{\binom{42}{29}}{\binom{45}{32}} \quad \begin{matrix} \to \frac{42!}{29!\,13!} \\ \to \frac{45!}{32!\,13!} \end{matrix}$$

$$= \frac{42!}{29!\,13!} \cdot \frac{32.31.30.29!\,.\,13!}{45.44.43.42!}$$

$$= \frac{32.31.30}{45.44.43}$$

$$\approx .35$$ //

▲ Exr. Textbook, page 71, Example 2.23.

2.4. Conditional Probability

Conditional probability provides us with a way to reason about the outcome of an experiment, based on partial information.

Given an experiment with sample space and probability law, suppose we know the outcome is in event B. We wish to find the probability that the outcome is also in event A. This is called the conditional probability of A given B, and is denoted by $P(A|B)$.

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The definition of conditional probability

Definition. For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has occured is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} . \qquad \ldots (*)$$

Example. Among buyers of a certain digital camera:

* 60% purchase an optinal memory card,
* 40% purchase an extra battery,
* 30% purchase both a memory card and a battery.

Let $A = \{$memory card purchased$\}$ and $B = \{$battery purchased$\}$. Then

a) If a buyer bought a battery, what is the probability they also bought a memory card?

b) If a buyer bought a memory card, what is the probability they also bought a battery?

Solution. We are given $P(A) = .6$, $P(B) = .4$, and $P(A \cap B) = .3$.

a) $P(\text{memory card} \mid \text{battery}) = P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.3}{.4} = .75$

b) $P(\text{battery} \mid \text{memory card}) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.3}{.6} = .5$

▲Exr. Textbook, page 76, Example 2.26.

The multiplication rule for $P(A \cap B)$

When we multiply both sides of eq $(*)$ by $P(B)$, we get the following result.

The multiplication rule. $P(A \cap B) = P(A|B) \cdot P(B)$,

SIMILARLY

$$P(A \cap B) = P(B|A) \cdot P(A).$$

Example. Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

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Solution. Let us define the events $A = \{$ first fuse is defective $\}$
$B = \{$ second fuse is defective $\}$.

We wish to find $P(A \cap B)$. By the multiplication rule,

$$P(A \cap B) = P(A)\,P(B|A) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21}$$

Here, $P(A) = \frac{N(A)}{N} = \frac{2}{7}$. Now, we are left with 6 total and 1 defective, which yields $P(B|A) = \frac{1}{6}$.

! The multiplication rule is most useful when the experiment consists of several stages in succession.

Extended multiplication rule

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 | A_1 \cap A_2)\,P(A_1 \cap A_2)$$
$$= P(A_3 | A_1 \cap A_2)\,P(A_2 | A_1)\,P(A_1),$$

where A_1 occurs first, followed by A_2, and then A_3.

Example. Suppose that five good and two defective fuses have been mixed up. To find the defective ones, we test them one by one, at random and without replacement. What is the probability that we find both of the defective fuses in exactly three tests?

Solution. Consider $A_1 = \{$ first fuse is defective $\}$, $B_1 = \{$ first fuse is good $\}$,
$A_2 = \{$ second fuse is defective $\}$, $B_2 = \{$ second fuse is good $\}$,
$A_3 = \{$ third fuse is defective $\}$, $B_3 = \{$ third fuse is good $\}$.

We are interested in $(B_1 \cap A_2 \cap A_3) \cup (A_1 \cap B_2 \cap A_3)$. Here "exactly three tests" means the second defective fuse must appear on the 3rd test. Hence,

$$P((B_1 \cap A_2 \cap A_3) \cup (A_1 \cap B_2 \cap A_3)) = P(B_1 \cap A_2 \cap A_3) + P(A_1 \cap B_2 \cap A_3)$$
$$= P(A_3 | B_1 \cap A_2)\,P(A_2 | B_1)\,P(B_1) + P(A_3 | A_1 \cap B_2)\,P(B_2 | A_1)\,P(A_1)$$
$$= \frac{1}{5} \cdot \frac{2}{6} \cdot \frac{5}{7} + \frac{1}{5} \cdot \frac{5}{6} \cdot \frac{2}{7}$$
$$= 2/21 \approx .095$$

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Bayes' theorem

Recall that events $A_1, \ldots, A_k$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for $i \neq j$.

The events are exhaustive if one A_i must occur, so that $A_1 \cup A_2 \cup \cdots \cup A_k = \Omega$.

Theorem:
(The law of total probability). Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k)$$
$$= \sum_{i=1}^{k} P(B|A_i)P(A_i).$$

Proof. Since $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive, if B occurs it provides $(B \cap A_i) \cap (B \cap A_j) \neq \emptyset$ for $i \neq j$. Hence $\{B \cap A_1, B \cap A_2, \ldots, B \cap A_k\}$ is a set of mutually exclusive events. Now

$$\Omega = A_1 \cup A_2 \cup \cdots \cup A_k$$

gives

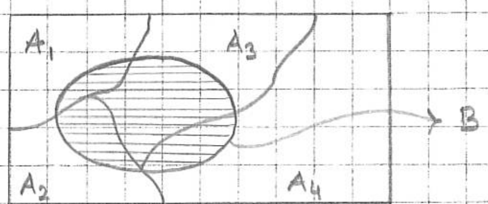
$$B = B \cap \Omega = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k). \text{ (Partitioning of B)}$$

Therefore,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_k).$$

But $P(B \cap A_i) = P(B|A_i)P(A_i)$ for $i = 1, 2, \ldots, k$, so

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k). \blacksquare$$



Partition of B by mutually exclusive and exhaustive A_i's

Theorem (Bayes' Theorem). Let $A_1, \ldots, A_k$ be a collection of mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, \ldots, k$. Then for any other event B, for which $P(B) > 0$

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}, \quad j = 1, \ldots, k.$$

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In statistical applications of Bayes' theorem $A_1, A_2, \ldots, A_k$ are called <u>hypotheses</u>, $P(A_i)$ is called the <u>prior</u> probability of A_i, and the conditional probability $P(A_j|B)$ is called the <u>posterior</u> probability of A_j after the occurrence of B.

<u>Example</u>. A box contains seven red and 13 blue balls. Two balls are selected at random and are discarded without their colors being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue?

<u>Solution</u>. • Discarded balls could be: RR, BB, and BR.

• Third ball: R

We wish to find $P(BB|R)$. By Bayes' theorem,

$$P(BB|R) = \frac{P(R|BB)\,P(BB)}{P(R|RR)P(RR) + P(R|BB)P(BB) + P(R|BR)P(BR)} \quad \ldots (1)$$

We have

$$P(RR) = \frac{7}{20} \cdot \frac{6}{19} = \frac{21}{190}, \quad P(BB) = \frac{13}{\cancel{20}_{5}} \cdot \frac{\cancel{12}^{3}}{19} = \frac{39}{95}, \quad P(BR) = P(B_1 \cap R_2) + P(R_1 \cap B_2)$$

$$= \frac{13}{20} \cdot \frac{7}{19} + \frac{7}{20} \cdot \frac{13}{19} = \frac{91}{190}. \quad \ldots (2)$$

and

$$P(R|RR) = \frac{5}{18}$$ (we discarded two red balls, we are left with five red) ...(3)

$$P(R|BB) = \frac{7}{18}$$ (we discarded two blue balls, we are left with seven red) ...(4)

$$P(R|BR) = \frac{6}{18}$$ (we discarded one blue and one red, we are left with six red) ...(5)

By substituting (2), (3), (4), and (5) in (1), we obtain

$$P(BB|R) = \frac{\frac{7}{18} \cdot \frac{39}{95}}{\frac{5}{18} \cdot \frac{21}{190} + \frac{7}{18} \cdot \frac{39}{95} + \frac{6}{18} \cdot \frac{91}{190}} \approx .46.$$

• Bayes' theorem is often used for <u>inference</u>. There are a number of "causes" that may result in a certain "effect". We observe the effect, and we wish to infer the cause.

2.5 Independence

We have introduced the conditional probability $P(A|B)$ to capture the partial information that event B provides about event A. An important case arises when knowing that B occured does not change the probability of A.

Definition. Two events A and B are independent if $P(A|B) = P(A)$ and are dependent otherwise.

In order to obtain $P(B|A)$, we consider the definition of conditional probability and the multiplication rule, and $P(A|B) = P(A)$. That is,

$$P(B|A) \underset{\text{by cond. prob.}}{=} \frac{P(A\cap B)}{P(A)} \underset{\text{by multp. rule}}{=} \frac{P(A|B).P(B)}{P(A)} \underset{\text{by } P(A|B)=P(A)}{=} P(B).$$

Example. Consider an ordinary deck of 52 cards comprised of the four "suits" spades, hearts, diamonds, and clubs, with each suit consisting of the 13 denominations ace, king, queen, jack, ten, ..., and two. Suppose someone randomly selects a card from the deck and reveals to you that it is a face card (a king, queen, or jack). What now is the probability that the card is a spade?

Solution. $A = \{\text{spade}\} \Rightarrow P(A) = \frac{13}{52}$

$B = \{\text{face card}\} \Rightarrow P(B) = \frac{12}{52}$

$A\cap B = \{\text{spade and face card}\} \Rightarrow P(A\cap B) = \frac{3}{52}$

By the definition of conditional probability,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{3/52}{12/52} = \frac{1}{4} = \frac{13}{52} = P(A)$$

This means that the likelihood of getting a spade is not affected by the knowledge of B has occured.

$P(A \cap B)$ when events are Independent

The next result shows how to find $P(A \cap B)$ when the events are independent.

Proposition. A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

In order to verify this proposition, we can consider the multiplication rule and the independence of A and B.

$$P(A \cap B) \underset{\text{multipl. rule}}{=} P(A|B) \cdot P(B) \underset{P(A|B) = P(A)}{=} P(A) \cdot P(B)$$

Example. Each day, Monday through Friday, a batch of components sent by a first supplier arrives at a certain inspection facility. Two days a week, a batch also arrives from a second supplier. Eighty percent of all supplier 1's bathches pass inspection, and 90% of supplier 2's do likewise. What is the probability that, on a randomly selected day, two batches pass inspection?

Solution. $A = \{$ two batches are received on the selected day $\} \Rightarrow P(A) = \frac{2}{5} = .4$

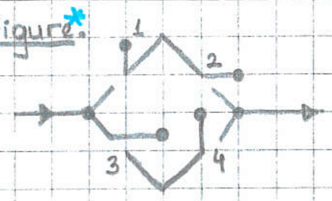
$B = \{$ both batches pass inspection $\} = (S_1 \cap S_2) \Rightarrow P(S_1 \cap S_2) = (.8)(.9) = .72$

We wish to find $P(A \cap B)$. By employing the cond. prob. definition,

$$\begin{aligned} P(A \cap B) &= P(B|A) \cdot P(A) \\ &= P(B) \cdot P(A) \\ &= P(S_1 \cap S_2) \cdot P(A) \\ &= (.72)(.4) = .288 \end{aligned}$$

$P(B|A) = P(B) \rightarrow$ definition of the independent events.

Figure*



Independence of more than two events

Definition. Events $A_1, \dots, A_n$ are mutually independent if for every k $(k = 2, \dots, n)$ and every subset of indices $i_1, i_2, \dots, i_k$, $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdot \dots \cdot P(A_{i_k})$.

Example. Figure* shows an electric circuit in which each of the switches located at 1, 2, 3, and 4 is independently closed or open with probabilities p and $1-p$, respectively. If a signal is fed to the input, what is the prob. that it is transmitted to the output?

$A_3 \cap A_4 = A_3A_4$

$A_1 \cap A_2 = A_1A_2$

Solution. $A_i = \{$ switch i is closed $\}$ for $i = 1, 2, 3, 4$. We wish to find $P((A_1 \cap A_2) \cup (A_3 \cap A_4))$.

$$\begin{aligned} \hookrightarrow P(A_1A_2 \cup A_3A_4) &= P(A_1A_2) + P(A_3A_4) - P(A_1A_2 \cap A_3A_4) \\ &= p \cdot p + p \cdot p - p \cdot p \cdot p \cdot p = p^2(2 - p^2). \end{aligned}$$